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Some open problems of generalised Bessel polynomials

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Abstract. The solution of several open problems connected with the generalised Bessel polynomials, which appear in solving the wave equation in spherical coordinates and in network synthesis and design, is shown. In particular, explicit and simple recursion formulae for sums of powers and product sums of the zeros of these polynomials are found. Also, three different sets of sums of generalised Bessel polynomials are analytically evaluated in a compact way.

1. Introduction

The Bessel polynomials appeared in the early thirties (Bochner 1929, Burchnall and Chaundy 1931) as the fourth class of orthogonal polynomials satisfying a second-order differential equation, the others being the classical systems of Jacobi, Laguerre and Hermite. However, the first systematic study of their properties was not done till twenty years later (Krall and Frink 1949) in connection with the solution of the wave equation in spherical coordinates. Shortly afterwards it was realised (Thomson 1949, 1952) the important role which these polynomials play in the theory of networks so that today they can be found not only in advanced articles (see e.g. Marshak *et al* 1974, Johnson *et al* 1976) but also in textbooks (Guillemin 1958, Hazony 1963, Weinberg 1975) of network synthesis and design. For more information and details about the Bessel polynomials and its applications see the excellent monograph (Grosswald 1978).

Here it is our purpose to show the solution of the following open problems of the Generalised Bessel Polynomials (GBP's) $y_n(x; a, b)$.

(i) To find explicit formulae for the Newton sums s_r, $r = 1, 2, \ldots$ of $y_n(x; a, b)$, that is for the rth power sum symmetric functions or just sums of rth powers of the zeros of the polynomial $y_n(x; a, b)$.

(ii) To find simple recurrence relations for the sums s_r.

(iii) To obtain explicit expressions for the so-called homogeneous product sum symmetric functions h_r, $r = 1, 2, \ldots$ of the zeros of the polynomial $y_n(x; a, b)$.

(iv) To derive new partial sums of GBP's in an analytical way.

The first two problems are explicitly pointed out by Grosswald (1978). They involve the quantities s_r which when conveniently normalised represent the moments about the origin of the distribution density of zeros of the polynomial $y_n(x; a, b)$.

The structure of the paper is as follows. In § 2 we briefly summarise the definition and the properties of the GBP's which are needed for our discussion. The following section contains the solutions and proofs of the first two problems, that is those which involve the most useful and elementary sum rules of the zeros of the polynomial $y_n(x; a, b)$. Section 4 is devoted to problem (iii), then to the more complicated sum rules of zeros h_r. Finally in § 5 problem (iv) is considered. In particular three different sets of formulae for some partial sums of GBP's are developed.

2. Review

The GBP $y_n(x; a, b)$ was defined (Krall and Frink 1949) as the polynomial solution of the differential equation

$$x^2y''+(ax+b)y'-n(n+a-1)y=0, \qquad b\neq 0,\ a\neq 0, -1, -2, \ldots . \tag{1}$$

Since $y_n(bx; a, b)$ is independent of b, it turns out that b is only a scale factor for the independent variable and not an essential parameter. This is why some authors prefer to use the polynomials $y_n(x; a)\equiv y_n(x; a, 2)$ or even $Y_n^{(\alpha)}(x)\equiv y_n(x; \alpha+2, 2)$ so that $y_n(x; 2)=Y_n^{(0)}(x)=y_n(x)$, the ordinary Bessel polynomial (Grosswald 1978, Chihara 1978).

The explicit expression of the GBP $y_n(x; a, b)$ is (Grosswald 1978)

$$y_n(x; a, b)=\sum_{i=0}^{n}\frac{b^i}{i!}\frac{n^{(i)}}{(2n+a-2)^{(i)}}x^{n-i}. \tag{2}$$

Here we have used the notation

$$u^{(i)}=u(u-1)(u-2)\ldots(u-i+1), \qquad i\geq 1,\ u^{(0)}\equiv 1.$$

In addition the GBP's satisfy the three-term recursion relation (Krall and Frink 1949)

$$\begin{aligned}(n+a-1)(2n+a-2)y_{n+1}&\\ =[(2n+a)(2n+a-2)(x/b)+a-2](2n+a-1)y_n&\\ +n(2n+a)y_{n-1}, \qquad n\geq 2,&\end{aligned} \tag{3}$$

with the initial conditions $y_0(x)=1$ and $y_1(x)=1+a(x/b)$.

3. Sum rules of zeros s_r

Let us denote by s_r the sums of the rth power of the zeros $\{x_1, x_2, \ldots, x_n\}$ of the polynomial $y_n(x; a, b)$, i.e.

$$s_r=\sum_{\nu=1}^{n}x_\nu^r, \qquad r=1, 2, \ldots .$$

The explicit expression of s_r in terms of a and b turns out to be

$$s_r=\sum_{(\lambda)}(-1)^{r-\Sigma\lambda}\frac{(-1+\Sigma\lambda)!\,r}{\lambda_1!\lambda_2!\ldots\lambda_n!}\prod_{i=1}^{n}\left(\frac{(-b)^i n^{(i)}}{i!(2n+a-2)^{(i)}}\right)^{\lambda_i} \tag{4}$$

where $\Sigma\lambda\equiv\Sigma_{i=1}^{n}\lambda_i$, and the summation $\Sigma_{(\lambda)}$ runs over all the partitions $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ of the number r so that $\Sigma_{i=1}^{n} i\lambda_i=r$.

Furthermore we will show that the quantities s_r satisfy the simple recurrence relation

$$\{2[n-\tfrac{1}{2}(r+2)]+a\}s_{r+1}=-\left(bs_r+\sum_{t=1}^{r} s_{r+1-t}s_t\right) \tag{5}$$

with the initial condition $s_1=-bn/(2n+a-2)$.

From equations (4) or (5), one can find a number of results encountered by different authors (Grosswald 1978). In particular the first few sums are

$$s_2=b^2n(n+a-2)/(2n+a-3)(2n+a-2)^2,$$

$$s_3=-b^3n(a-2)(n+a-2)/(2n+a-2)^3(2n+a-3)(2n+a-4),$$

$$s_4=\frac{b^4n(n+a-2)}{(2n+a-2)^4(2n+a-3)^2(2n+a-4)}[(a^3-7a^2+16a-12)+(a^2-2a)n$$
$$+(8-3a)n^2-2n^3].$$

From (4) one can easily see that, for ordinary Bessel polynomials, the sum $s_{2k+1}=0$ for $k\geq 1$ as was pointed out by Ismail and Kelker (1976). Notice from the same equation that the quantities s_r/n for $n\to\infty$ and $r=1,2,\ldots$ vanish as it is also known (Dehesa 1978).

To prove (3) we will use the following result (Raghavacharyulu and Tekumalla 1972).

Let $P_n(x)$ be the polynomial

$$P_n(x)=\sum_{i=0}^{n}(-1)^i a_i x^{n-i}, \qquad \text{with } a_0=1. \tag{6}$$

then

$$s_r=\sum_{(\lambda)}(-1)^{r-\Sigma\lambda}(-1+\Sigma\lambda)!\,r\prod_{i=1}^{n}\left(\frac{a_i^{\lambda_i}}{\lambda_i!}\right) \tag{7}$$

where all the symbols are as before. The comparison between (2) and (6) allows one to write

$$a_i=\frac{(-b)^i}{i!}\frac{n^{(i)}}{(2n+a-2)^{(i)}}. \tag{8}$$

Taking this value to equation (6) one immediately finds the required equation (4).

To prove (5) we will use a different procedure. We will not start from the explicit expression of the polynomials but from the differential equation fulfilled by them and the following general result (Case 1980) will be used. Let us assume the polynomials $P_n(x)$ satisfy a second-order differential equation of the form

$$g_2(x)P_n''(x)+g_1(x)P_n'(x)+g_0(x)P_n(x)=0 \tag{9}$$

where

$$g_i(x)=\sum_{j=0}^{i}a_j^{(i)}x^j, \qquad i=0,1,2,\ldots \tag{10}$$

with constant coefficients $a_j^{(i)}$. Assuming further that the zeros of $P_n(x)$ are simple, then the recurrence relation is fulfilled

$$2[a_0^{(2)}J_r+a_1^{(2)}J_{r+1}+a_2^{(2)}J_{r+2}]=-a_0^{(1)}s_r-a_1^{(1)}s_{r+1}, \qquad r=0,1,2,\ldots \tag{11}$$

with the initial condition $s_0 = N$ and where

$$J_k = \sum_{l_1 \neq l_2} x_{l_1}^k/(x_{l_1} - x_{l_2})$$
$$= \begin{cases} 0, & k=0, \\ n(n-1)/2, & k=1, \\ (n-1)s_1, & k=2, \\ (n-k/2)s_{k-1} + \frac{1}{2}\sum_{t=1}^{k-2} s_{k-1-t}s_t, & k \geq 3. \end{cases} \qquad (12)$$

The comparison between (1) and (9) gives

$$a_2^{(2)} = 1, \qquad a_1^{(2)} = 0, \qquad a_0^{(2)} = 0,$$
$$a_1^{(1)} = a, \qquad a_0^{(1)} = b, \qquad a_0^{(0)} = -n(n+a-1).$$

For these values, the basic relation (11) reduces as

$$2J_{r+2} = -bs_r - as_{r+1}, \qquad r = 0, 1, 2, \ldots .$$

Taking into account (12), one observes that this relation leads in a straightforward manner to the required equation (5).

Finally let us mention that another new but more complicated recurrence relation for the sums s_r can be easily obtained by making the observation that (Raghavacharyulu and Tekumalla 1972)

$$s_r = V_r^{(n)}(a_1, a_2, \ldots, a_n)$$

and that (Riordan 1958)

$$s_r = -\frac{1}{(r-1)!} Y_r(-f_1 a_1, f_2 2!\, a_2, -f_3 3!\, a_3, \ldots, f_r(-1)^r r!\, a_r)$$
$$f_j = (-1)^{j-1}(j-1)!$$

for an arbitrary polynomial written in the form (6). Here V_r and Y_r are so-called generalised Lucas polynomials of second type and the well known Bell polynomials of the number theory respectively. The recurrence properties of these polynomials together with the values (8) of a_i also supply useful recursion relations for the Newton sums of the GBP's.

4. Additional sum rules of zeros h_r

Let us consider an arbitrary polynomial $P_n(x)$ in the form (6). The so-called homogeneous product sums symmetric functions $h_r \equiv h_r(x_1, x_2, \ldots, x_n)$ of the zeros of this polynomial are defined as follows (Riordan 1958)

$$\prod_{i=1}^{n} (1 - x_i x) = (1 + h_1 x + h_2 x^2 + h_3 x^3 + \ldots)^{-1}.$$

The first few sums are given explicitly by

$$h_1 = \sum_{i=1}^{n} x_i,$$

$$h_2 = \sum_{i=1}^{n} x_i^2 + \sum_{i<j}^{n} x_i x_j,$$

$$h_3 = \sum_{i=1}^{n} x_i^3 + \sum_{i \neq j} x_i^2 x_j + \sum_{i<j<k} x_i x_j x_k.$$

Here we will show that the product sums h_r of the zeros of the GBP $y_n(x; a, b)$ have the explicit formula

$$h_r = \sum_{(\lambda)} (-1)^{r-\Sigma\lambda} \frac{(\Sigma\lambda)!}{\lambda_1!\lambda_2!\ldots\lambda_n!} \prod_{i=1}^{n} \left[\frac{(-b)^i n^{(i)}}{i!(2n+a-2)^{(i)}}\right]^{\lambda_i} \tag{13}$$

where the symbols are the same as in (4). The first three sums are

$$h_1 = \frac{-bn}{(2n+a-2)},$$

$$h_2 = b^2 n[n(2n+a-2)+a-2]/2(2n+a-2)^2(2n+a-3).$$

$$h_3 = -\frac{b^3 n[4n^4+4(a-2)n^3+(a^2+2a-12)n^2+(3a^2-16a+20)n+2(a-2)^2]}{6(2n+a-2)^3(2n+a-3)(2n+a-4)}.$$

Let us prove it. Since

$$1 - a_1 x + a_2 x^2 - a_3 x^3 + \ldots = (1 + h_1 x + h_2 x^2 + h_3 x^3 + \ldots)^{-1}$$

it can be shown that the product sums h_r and the elementary symmetric functions a_s, $s = 1, 2, \ldots$ are related by

$$h_r = \sum_{(\lambda)} (-1)^{r-\Sigma\lambda} \frac{(\Sigma\lambda)!}{\lambda_1!\lambda_2!\ldots\lambda_n!} a_1^{\lambda_1} a_2^{\lambda_2} \ldots a_n^{\lambda_n} \tag{14}$$

where all the symbols are already known. To obtain (13) from (14) it is enough to take into account the values (8) of the a_i of the GBPS.

To end this section it is interesting to remark that the product sums h_r can be expressed in terms of the generalised Lucas polynomials of the first type (Raghavacharyulu and Tekumalla 1972) as

$$h_r = U_{r+n-1}^{(n)}(a_1, a_2, \ldots, a_n)$$

and in terms of the Bell polynomials (Lavoie 1975) as

$$h_r = \frac{1}{r!} Y_r(f_1 g_1, f_2 g_2, \ldots, f_r g_r)$$

with $f_i = (-1)^i i!$ and $g_i = f_i a_i$ for $i \leq n$; $g_i = 0$ for $i > n$. The use of the known recursion relations of the U-polynomials (Raghavachrayulu and Tekumalla 1972, Williams 1971) and of the Bell polynomials (Riordan 1958) together with (8) easily allows one to find recurrence formulae for the symmetric functions h_r of the zeros of the GBPS.

5. Sums of Bessel polynomials

When expanding a given function in terms of a system of orthogonal polynomials $\{\pi_k(x); k=0, 1, 2, \ldots\}$ the question arises of how to evaluate partial sums of the type $\Sigma\, c_k\pi_k(x)$. Numerically this can be done by means of the Clenshaw's algorithm (Ng 1968). However analytical expressions for such sums do not exist except for some classical sets of orthogonal polynomials (Hansen 1981).

Here we will show three different sets of certain sums of GBP's the analytical solution of which are especially compact and simple. They are written in the form of three theorems. Some corollaries and proofs follow.

Theorem 1. The GBP's satisfy

$$\sum_{n=k}^{m-1} (-1)^{n+1}\frac{(2n+a)^{(2n+1)}(n+a)}{(n+1)!(2n+a+2)} A'_n y_n(x; a, b)$$
$$= \frac{(-1)^m}{m!}(2m+a-1)^{(2m)}A'_m y_m(x; a, b) - \frac{(-1)^k}{k!}(2k+a-1)^{(2k)}A'_k y_k(x; a, b) \tag{15a}$$

with

$$A'_\nu = \prod_{r=0}^{\nu-1}\left[\frac{x}{b}+\frac{a-2}{(2r+a)(2r+a+2)}\right], \qquad \nu = k, n, m. \tag{15b}$$

Corollary. The ordinary Bessel polynomial $y_n(x) \equiv y_n(x; 2, 2)$ verify

$$\sum_{n=k}^{m-1} (-1)^{n+1}(2n+1)!!\,x^n y_{n+2}(x) = (-1)^m(2m+1)!!\,y_m(x) - (-1)^k(2k+1)!!\,y_k(x) \tag{16}$$

where

$$u!! = u(u-2)(u-4)\ldots 1.$$

Theorem 2. The GBP's verify the relation

$$\sum_{n=k}^{m-1} \frac{n(n+a-2)^{(n-k)}}{(2n+a-1)^{(2n-2k+1)}(2n+a-2)} B'_n y_{n-1}(x; a, b)$$
$$= \frac{(m+a-2)^{(m-k)}}{(2m+a-2)^{(2m-2k)}} B'_{m-1} y_m(x; a, b) - y_k(x; a, b), \tag{17a}$$

with

$$B'_\nu = \prod_{r=k}^{\nu}\left(\frac{x}{b}+\frac{a-2}{(2r+a)(2r+a-2)}\right)^{-1}. \tag{17b}$$

Corollary. The ordinary Bessel polynomials $y_n(x)$ satisfy

$$\sum_{n=k}^{m-1} \frac{(2k-1)!!}{(2n+1)!!}\frac{y_{n-1}(x)}{x^{n-k+1}} = \frac{(2k-1)!!}{(2m-1)!!}\frac{y_m(x)}{x^{m-k}} - y_k(x). \tag{18}$$

Theorem 3. The GBP's have the following property

$$\sum_{n=k}^{m-1} \frac{(4n+2s+a+1)(4n+2s+a)}{2n+s+1}\left(\frac{x}{b}+\frac{a-2}{(4n+2s+a)(4n+2s+a+2)}\right) C'_n y_{2n+s+1}^{(x;a,b)}$$
$$= C'_m y_{2m+s}(x;a,b) - C'_k y_{2k+s}(x;\, a,\, b), \tag{19a}$$

with

$$C'_\nu = \frac{[\frac{1}{2}(s+a)+1]_{\nu-1}[\frac{1}{4}(2s+a)+1]_{\nu-1}}{[\frac{1}{2}(s+1)+1]_{\nu-1}[\frac{1}{4}(2s+a-2)+1]_{\nu-1}} \tag{19b}$$

and $s=0$ or 1.

Corollary. The ordinary Bessel polynomials verify the following relation

$$\sum_{n=k}^{m-1} (4n+2s+3)xy_{2n+s+1}(x) = y_{2m+s}(x) - y_{2k+s}(x). \tag{20}$$

Proof. To prove these theorems we will use the following result (Hansen 1981). Let us consider the recursion relation

$$a_n f_n(x) + b_n f_{n+1}(x) + c_n f_{n+2}(x) = 0 \tag{21}$$

for some function $f_n(x)$ where n takes integer values and the coefficients a_n, b_n, c_n may also depend on x. The three following sum rules for the functions $f_n(x)$ are fulfilled.

(1)

$$\sum_{n=k}^{m-1} \frac{c_n}{a_n} A_n f_{n+2}(x) = A_m f_m(x) - A_k f_k(x) \tag{22a}$$

with

$$A_r = \prod_{n=0}^{r-1} \left(\frac{-b_n}{a_n}\right). \tag{22b}$$

(2)

$$\sum_{n=k}^{m-1} \frac{a_{n-1}}{b_{n-1}} B_n f_{n-1}(x) = B_m f_m(x) - B_k f_k(x) \tag{23a}$$

with

$$B_r = \prod_{n=1}^{r-1} \left(\frac{-c_{n-1}}{b_{n-1}}\right). \tag{23b}$$

(3)

$$\sum_{n=k}^{m-1} \frac{b_{2n+s}}{a_{2n+s}} C_{2n+s}^{n} f_{2n+s+1}(x) = C_{2m+s} f_{2m+s}(x) - C_{2k+s} f_{2k+s} \tag{24a}$$

with $s=0$ or 1, and

$$C_{2r+s} = \prod_{n=0}^{r-1} \left(\frac{-c_{2n+s}}{a_{2n+s}}\right). \tag{24b}$$

The comparison between ($3a$) and (21) gives

$$a_n = (n+1)(2n+a+2), \tag{25a}$$

$$b_n = (2n+a+1)[(2n+a)(2n+a+2)(x/b)+a-2], \tag{25b}$$

$$c_n = -(n+a)(2n+a). \tag{25c}$$

One observes that

$$A_r = \frac{(-1)^r}{r!}(2r+a-1)^{(2r)} \prod_{n=0}^{r-1}\left[\frac{x}{b}+\frac{a-2}{(2n+a)(2n+a+2)}\right], \tag{26}$$

$$B_r = \frac{(r+a-2)^{(r)}}{(2r+a-2)^{(2r-2)}} \prod_{n=1}^{r-1}\left[\frac{x}{b}+\frac{a-2}{(2n+a)(2n+a-2)}\right]^{-1}, \tag{27}$$

$$C_{2r+s} = \frac{(s+a)(2s+a)}{(s+1)(2s+a+2)} \cdot \frac{[\frac{1}{2}(s+a)+1]_{r-1}[\frac{1}{4}(2s+a)+1]_{r-1}}{[\frac{1}{2}(s+1)+1]_{r-1}[\frac{1}{4}(2s+a+2)+1]_{r-1}}, \tag{28}$$

where Pochhammer's symbol $(z)_n = z(z+1)\ldots(z+n-1)$ has been used.

Taking the values ($25a$, c) and (26) to ($22a$) one easily obtains ($15a$, b) of theorem 1. Equation (16) of the corollary is found by replacing $a=b=2$ in ($15a$). In an analogous way the relations ($23a$, b) together with ($25a$, b) and (27) lead to ($17a$, b) and (18) of theorem 2. Also ($19a$, b) and (20) are obtained from ($24a$, b) together with ($25a$, b) and (28).

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